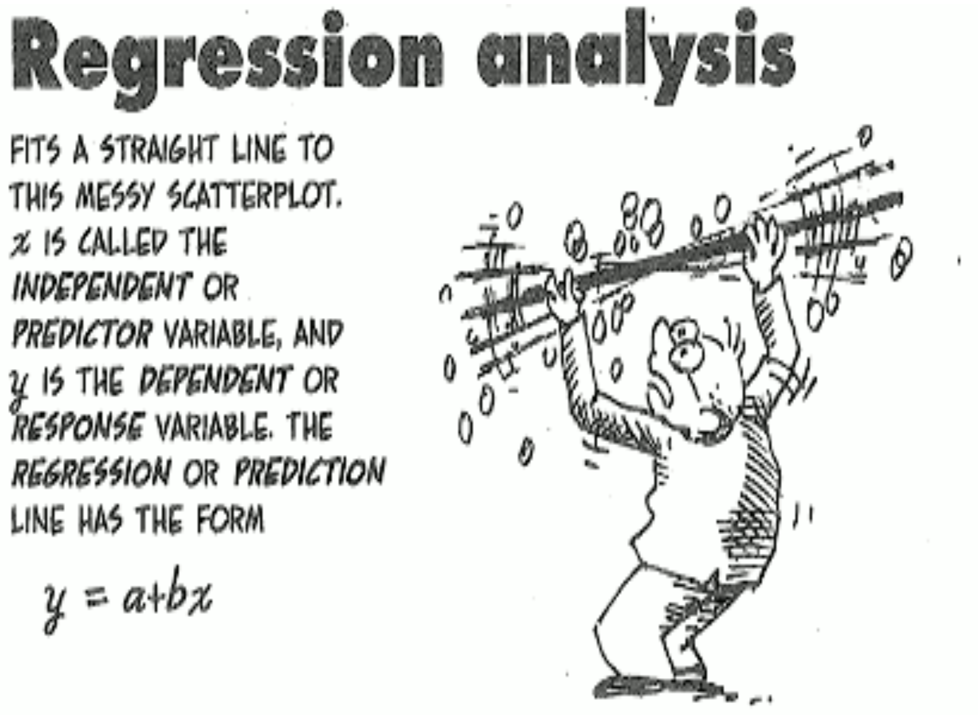
“This is statistics”

by Dr. Genevera Allen

Associate Professor at Rice University

https://www.youtube.com/watch?v=xURkTKtDq_M



Multiple Regression Analysis

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General linear models (not Generalized linear model)

Linear Model	Common name
$Y = \mu + X$	Simple linear regression
$Y = \mu + A_1$	One-factorial (one-way) ANOVA
$Y = \mu + A_1 + A_2 + A_1 \times A_2$	Two-factorial (two-way) ANOVA
$Y = \mu + A_1 + X\ (+A_1 \times X)$	Analysis of Covariance (ANCOVA)
$Y = \mu + X_1 + X_2 + X_3$	Multiple regression
$Y = \mu + A_1 + g + A_1 \times g$	Mixed model ANOVA
$Y_1 + Y_2 = \mu + A_1 + A_2 + A_1 \times A_2$	Multivariate ANOVA (MANOVA)

Y (response) is a continuous variable
X (predictor) is a continuous variable
A represents categorical predictors (factors)
g represents groups of data (more on this later)
$(+A_1 \times X)$ - step 1 on an ANCOVA, but not in the final analysis
Multiple factors $A_1 + A_2 + \text{etc (and their interactions)}$

Multiple regression – the "model of all models"!

Part I:

Causation, regression model, properties of estimators and sensibility to assumptions

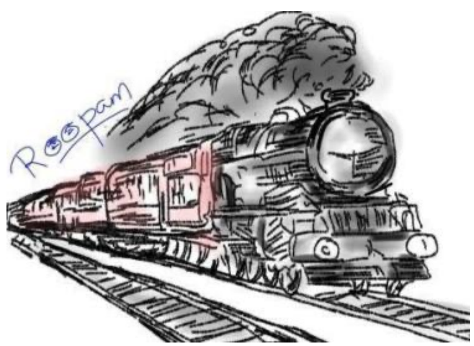
Part II:

Goodness of fit and model simplicity metrics, hypotheses testing, standardized slopes, model selection, examples and diagnostics

Multiple regression – the "model of all models"!

The essential idea with regression models is to find driving forces like the train engine and determine the path of the railway track.

The "driving force" in statistics is often called "generating process"



Correlation, Causation, & Coincidence

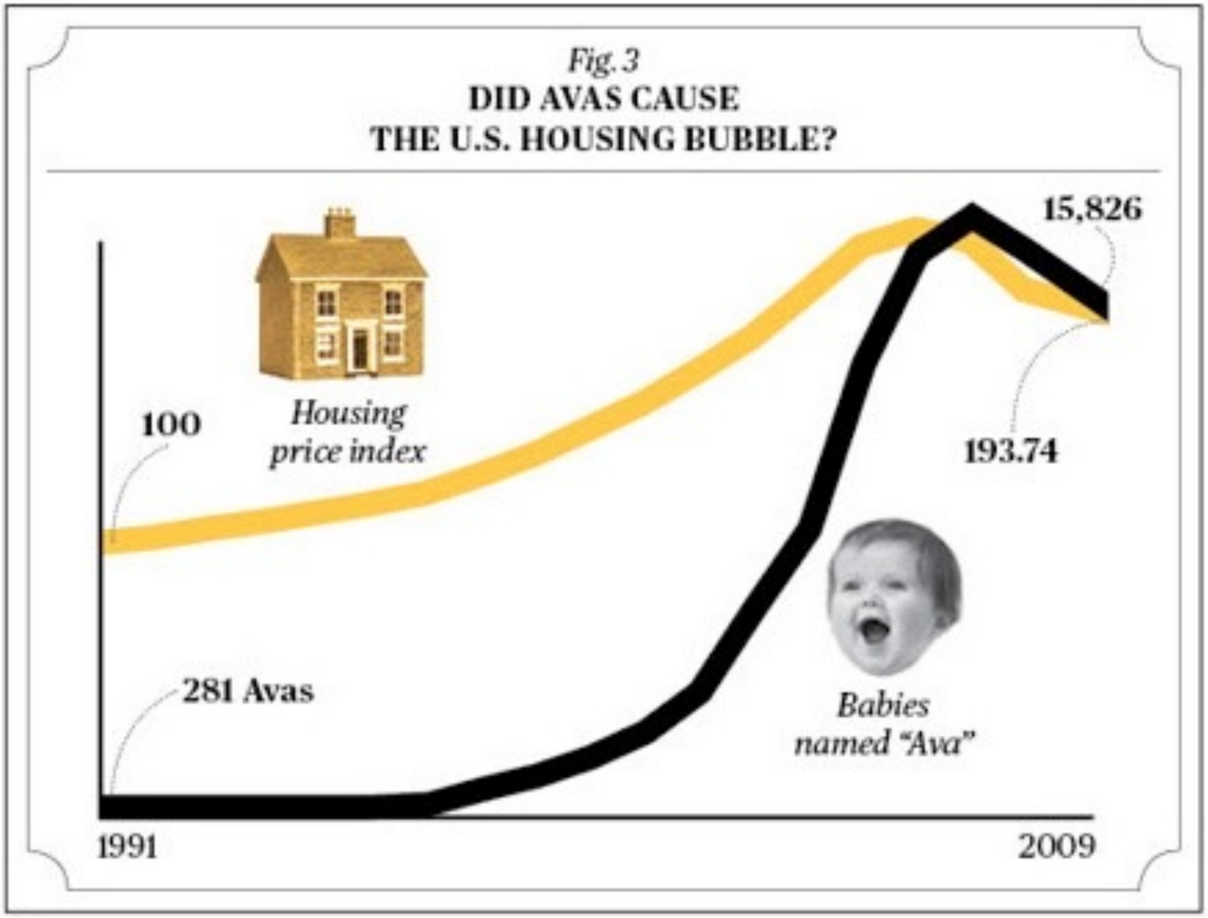
One of the key concepts in regression models, or science in general, is to distinguish between correlation and causation.

source - http://ucanalytics.com/

Unless in experimental settings and in some time series (and even then), regression models cannot necessarily distinguish between causation and correlation.

The role of researchers when using regression is to provide strong evidence and a narrative of causation (even though it can’t always be confirmed).

Likely a coincidence



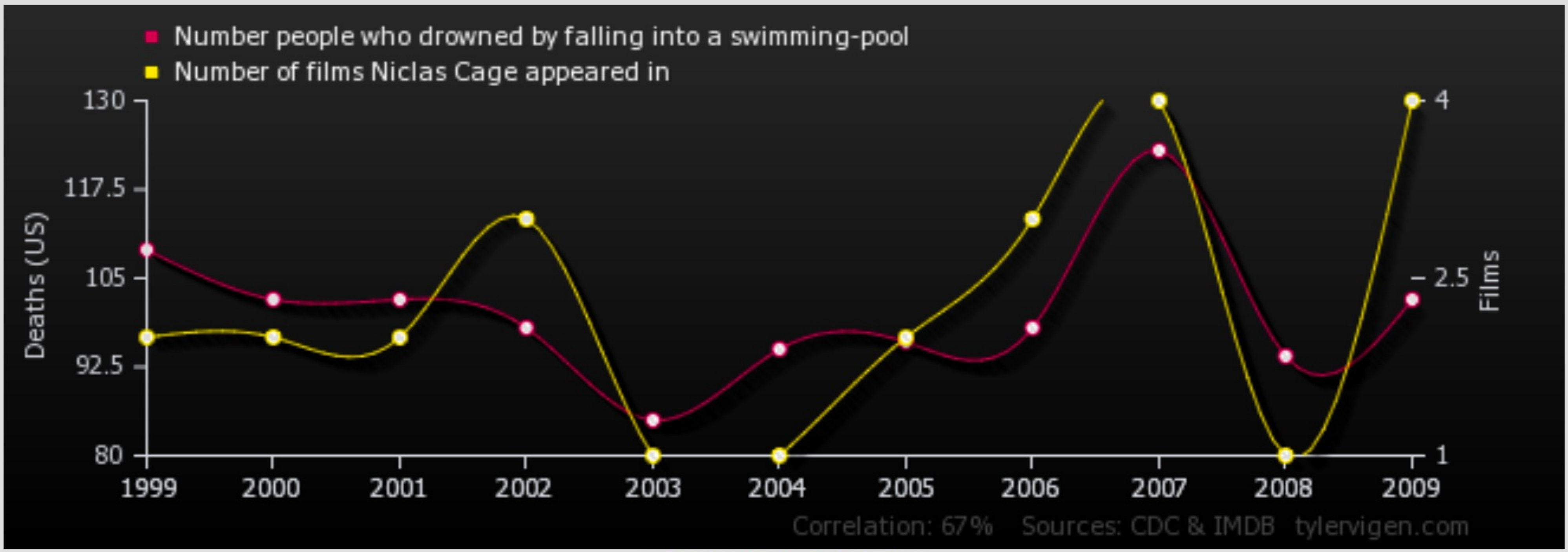
Source: Bloomberg

Likely a coincidence

Number people who drowned by falling into a swimming-pool

correlates with

Number of films Nicolas Cage appeared in

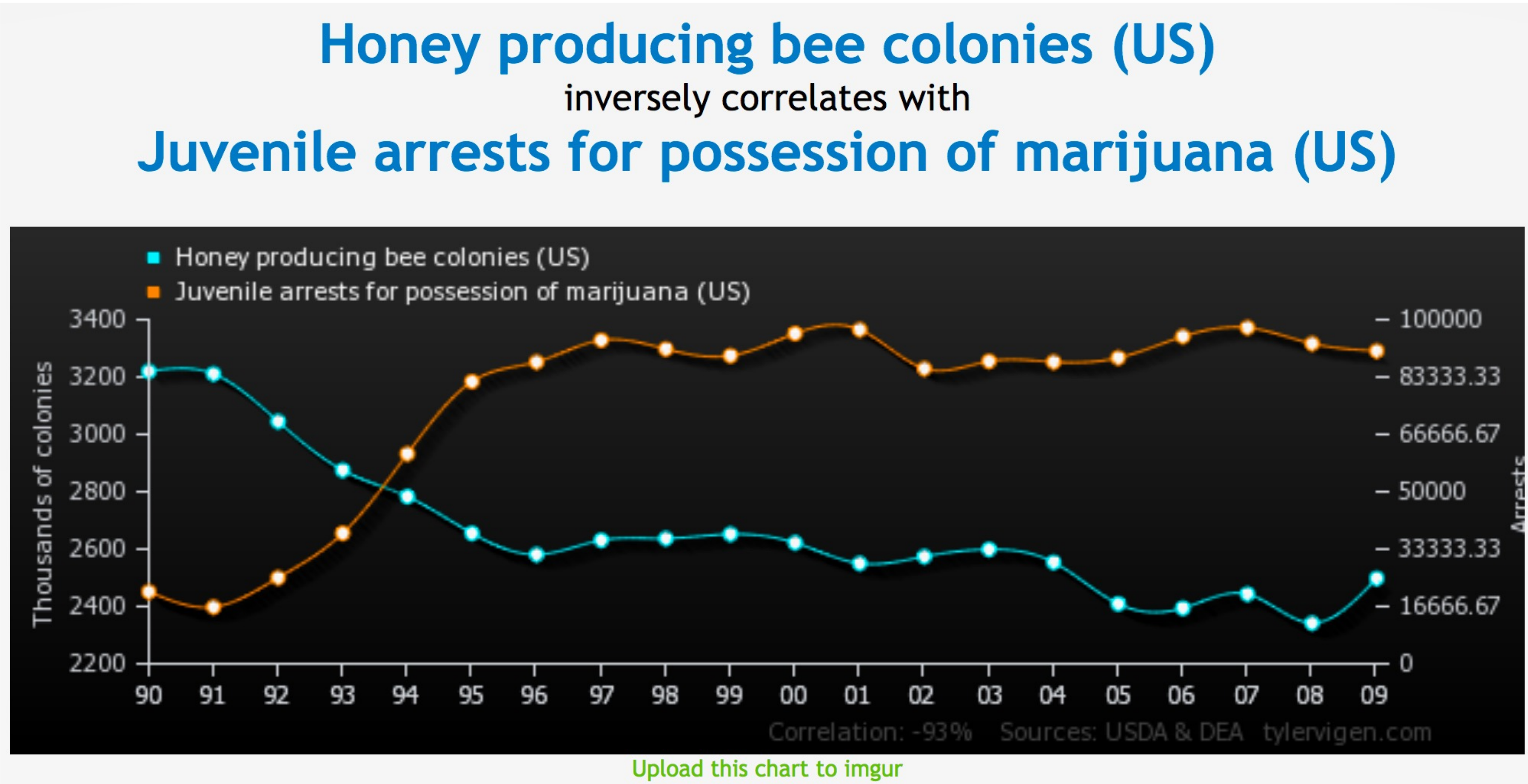


Upload this chart to imgur

Correlation: 0.666004

Source: http://tylervigen.com

Likely a coincidence



Correlation: -0.933389

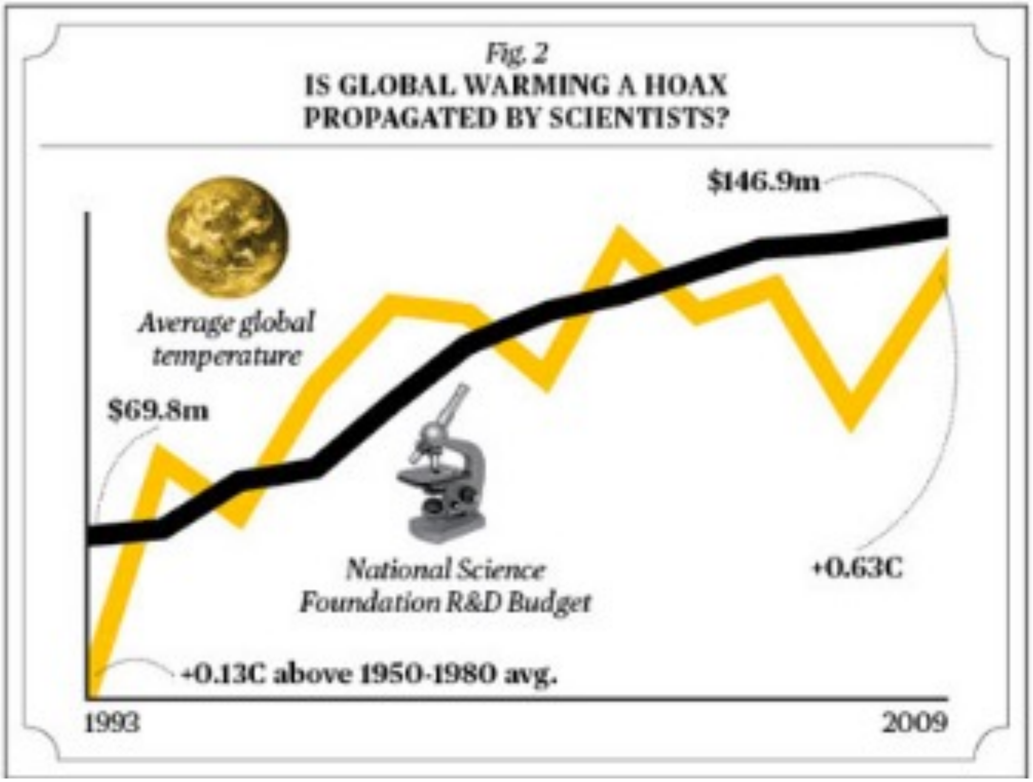
Source: http://tylervigen.com

Coincidence =

spurious correlations

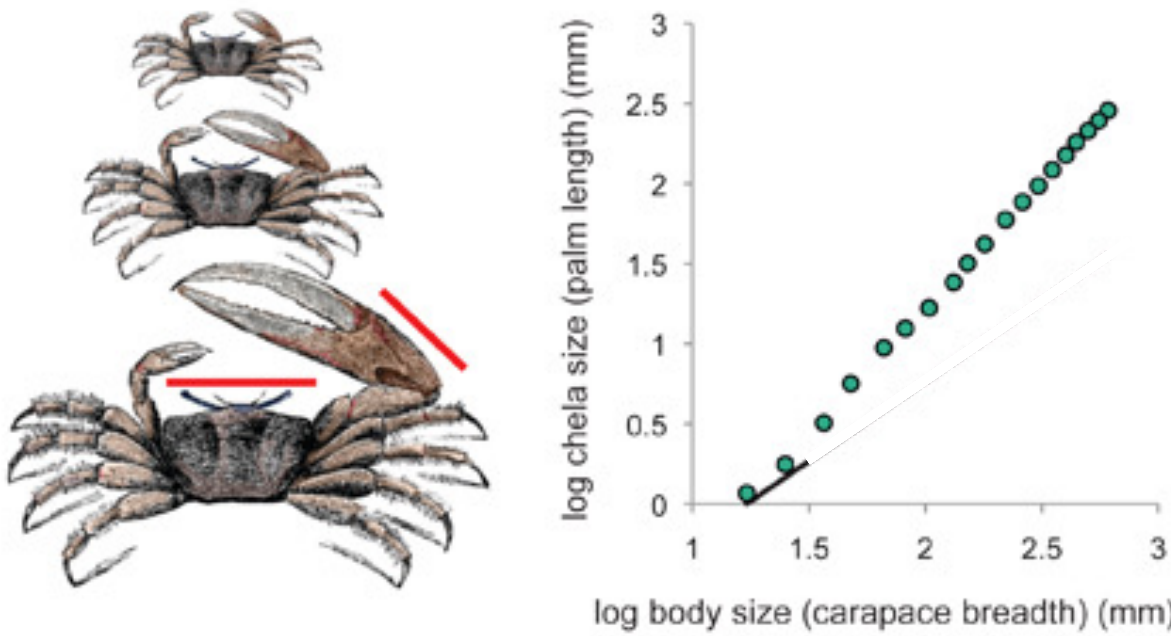
http://tylervigen.com/discover?type_select=fun

Likely a correlation



Source: Bloomberg

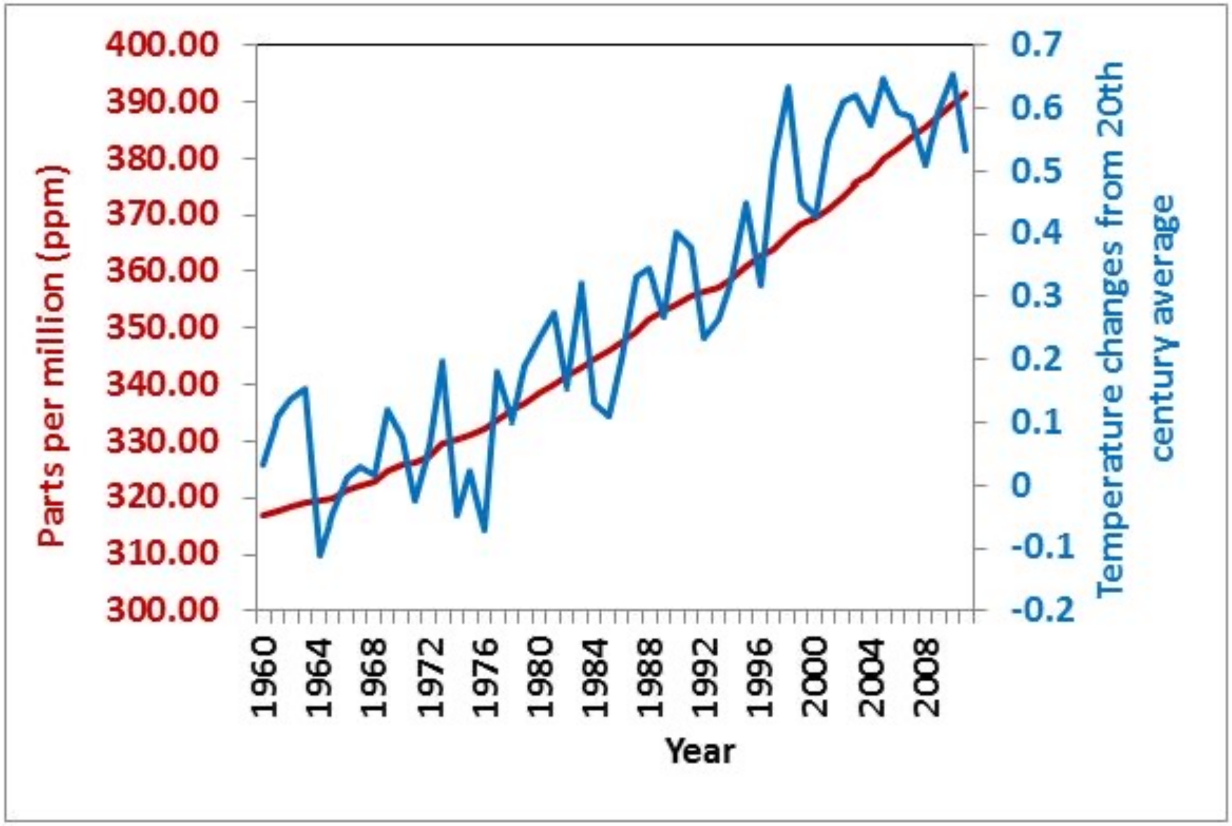
Correlation



Source: Nature

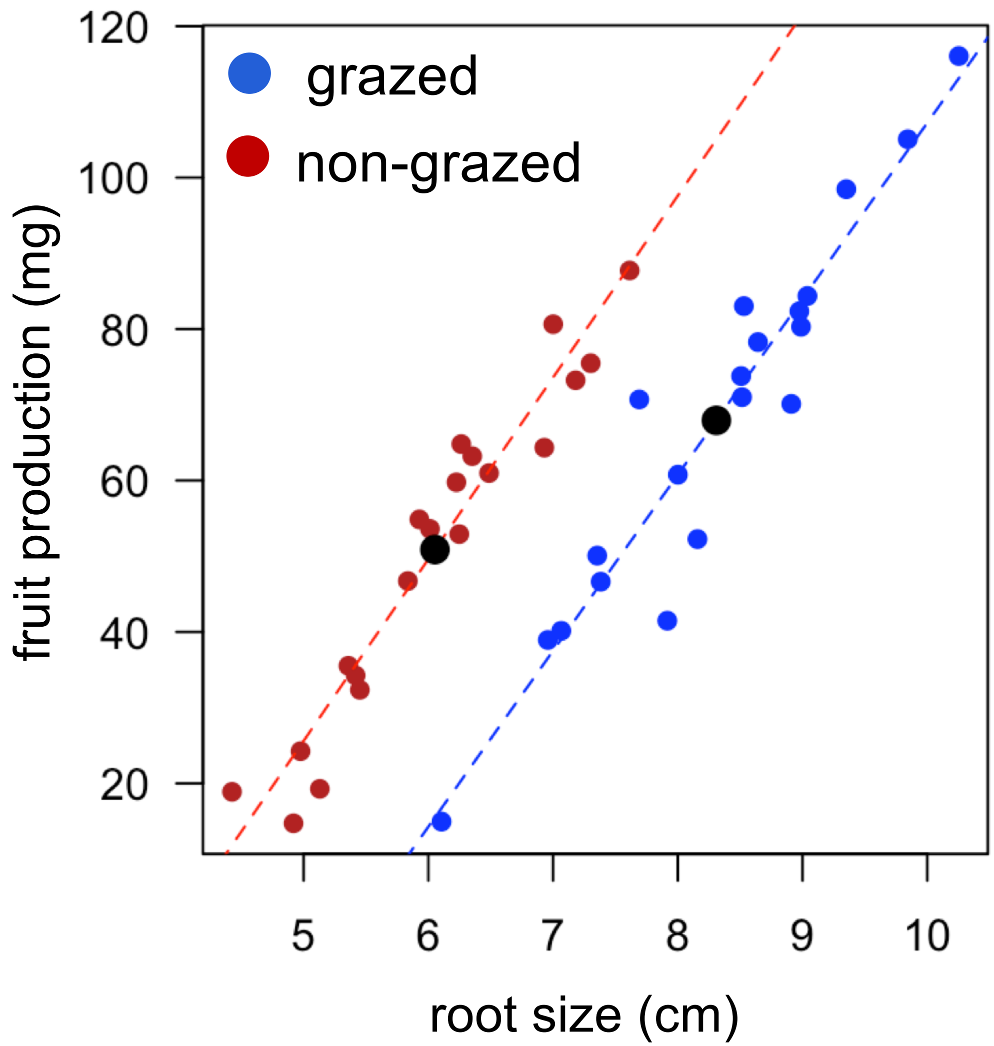
Causation

CO_2 concentration versus temperature



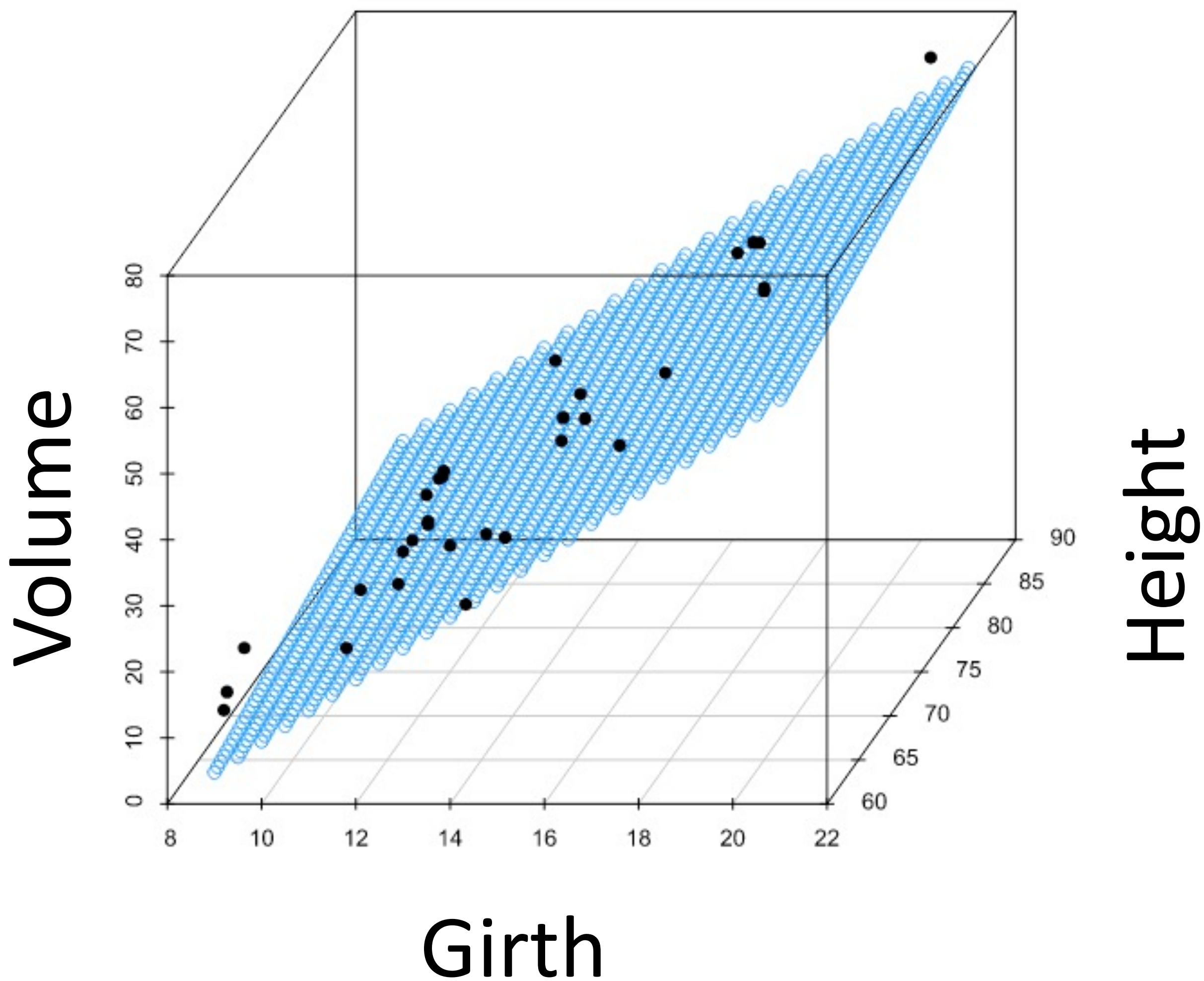
Source - www.e-education.psu.edu

Causation



grazed
non-grazed
120
100
80
60
40
20
fruit production (mg)
5
6
7
8
9
10
root size (cm)

Multiple regression (Cherry tree)



Discussion: Causation & Correlation *versus* Prediction

Some thoughts on « explanation »

In 1964, during a lecture at Cornell University, the physicist Richard Feynman articulated a profound mystery about the physical world. He told his listeners to imagine two objects, each gravitationally attracted to the other. How, he asked, should we predict their movements? Feynman identified three approaches, each invoking a different belief about the world.

source – The New Yorker

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1) The first approach used Newton's law of gravity, according to which the objects exert a pull on each other.

2) The second imagined a gravitational field extending through space, which the objects distort.

3) The third applied the principle of least action, which holds that each object moves by following the path that takes the least energy in the least time.

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2) The second imagined a gravitational field extending through space, which the objects distort.

3) The third applied the principle of least action, which holds that each object moves by following the path that takes the least energy in the least time.

All three approaches produced the same, correct prediction. They were three equally useful descriptions of how gravity works. "One of the amazing characteristics of nature is this variety of interpretational schemes," Feynman said.

source – The New Yorker

Multiple regression – the “models of all models”!

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_p X_p + e$$

β_0 model intercept (or constant)

$\beta_1, \beta_2, \ldots, \beta_p$ Partial regression coefficients (or partial slopes)

e model residuals or error

The general purpose of *multiple regression* are:

1) Describe, investigate and learn about the relationship between several independent or predictor variables and a dependent variable.

2) Make predictions.

3) Plan experiments to test causality (in regression, causality is often implied).

Multiple regression – the "models of all models"!

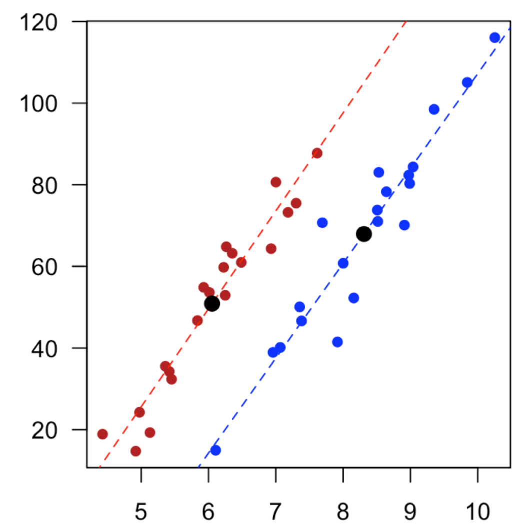
$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \dots + \beta_p X_p + e$$

β_0 model intercept (or constant)

$\beta_1, \beta_2, \dots, \beta_p$ Partial regression coefficients (or partial slopes)

e model residuals or error

Fitting method = Ordinary least square (OLS)



The OLS method minimizes the sum of square differences between the observed and predicted values.

A small fictional example to facilitate understanding of what regression coefficients mean!

$$Y = \mathbf{42cm} + \beta_1 X_1 + \beta_2 X_2 + e$$

Y is plant height (cm)
X_1 is amount of bacteria in the soil (1000 bacteria per ml of soil)
X_2 is amount of plant exposure to sun light (% exposure)

$$\beta_0$$

- Model intercept (or constant) is the value that is predicted for Y if predictors X_1 and X_2 are zero, i.e., the expected plant height if there is no bacteria in the soil and no sun light.

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- This is only a reasonable interpretation if either X_1 and X_2 can be zero and if the data include values for X_1 and X_2 that are closer to zero). For instance, the intercept could be negative for this model even though a plant can't have negative height.

- The unit of the intercept is the same as the response variable (i.e., cm).

A small fictional example to facilitate understanding of what regression coefficients mean!

$$Y = 42\text{cm} + \mathbf{2.3}X_1 + \beta_2 X_2 + e$$

Y is plant height (cm)
X_1 is amount of bacteria in the soil (1000 bacteria per ml of soil)
X_2 is amount of plant exposure to sun light (% exposure)

β_1

- It represents the difference in predicted value of Y (plant height) for each one unit difference in bacteria amount if sun exposure is kept constant (i.e., as if plants were exposed to the same amount of mean sun light) – called partial effects/slopes

- Plants with 5000/ml bacteria counts would, on average, be 2.3 cm taller (in average) than plants in soils with 4000/ml (which would be 2.3 cm taller in average than plants with 3000/ml).

The slope of any single partial regression line (partial regression slope) represents the rate of change or effect of that specific predictor variable (holding all the other predictor variables constant to their respective mean values) on the response variable.

A small fictional example to facilitate understanding of what regression coefficients mean!

$$Y = 42\text{cm} + \mathbf{2.3}\text{X}_1 + \beta_2\text{X}_2 + e$$

Y is plant height (cm)
X_1 is amount of bacteria in the soil (1000 bacteria per ml of soil)
X_2 is amount of plant exposure to sun light (% exposure)

$$\beta_1$$

Represents the difference in predicted value of Y (plant height) for each one unit difference in bacteria amount if sun exposure is kept constant (i.e., as if plants were exposed to the same mean amount of sun light).

A small fictional example to facilitate understanding of what regression coefficients mean!

$$Y = 42\text{cm} + \mathbf{2.3}X_1 + \beta_2 X_2 + e$$

Y is plant height (cm)
X_1 is amount of bacteria in the soil (1000 bacteria per ml of soil)
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β_1

- It represents the difference in predicted value of Y (plant height) for each one unit difference in bacteria amount if sun exposure is **kept constant** (i.e., as if plants were exposed to the same amount of sun light).

- Plants with 5000/ml bacteria counts would, on average, be 2.3 cm taller (in average) than plants in soils with 4000/ml (which would be 2.3 cm taller in average than plants with 3000/ml).

- "**Kept constant**" means that that the association between bacterial amount and plant height is independent (controlled for) of amount of sun.

A small fictional example to facilitate understanding of what regression coefficients mean!

$$Y = 42\text{cm} + \mathbf{2.3}X_1 + \beta_2 X_2 + e$$

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- Plants with 5000/ml bacteria counts would, on average, be 2.3 cm taller (in average) than plants in soils with 4000/ml (which would be 2.3 cm taller in average than plants with 3000/ml).
- **"Kept constant"** means that that the association between bacterial amount and plant height is independent (controlled for) of amount of sun.
- The unit attached to the slope is the unit of the response divided by the unit of the predictor (i.e., cm/ 1000 bacteria per ml)

A small fictional example to facilitate understanding of what regression coefficients mean!

$$Y = 42\text{cm} + \mathbf{2.3}X_1 + \mathbf{11}X_2 + e$$

Y is plant height (cm)
X_1 is amount of bacteria in the soil (1000 bacteria per ml of soil)
X_2 is amount of plant exposure to sun light (% exposure)

β_1

- It represents the difference in predicted value of Y (plant height) for each one unit difference in bacteria amount if amount of sun is kept constant (i.e., as if plants were exposed to the same amount of sun light).
- Plants with 5000/ml bacteria counts would, on average, be 2.3 cm taller (in average) than plants in soils with 4000/ml (which would be 2.3 cm taller in average than plants with 3000/ml).
- "Kept constant" means that that the association between bacterial amount and plant height is independent (controlled for) of amount of sun.

β_2

Reverse interpretation in relation to β_1

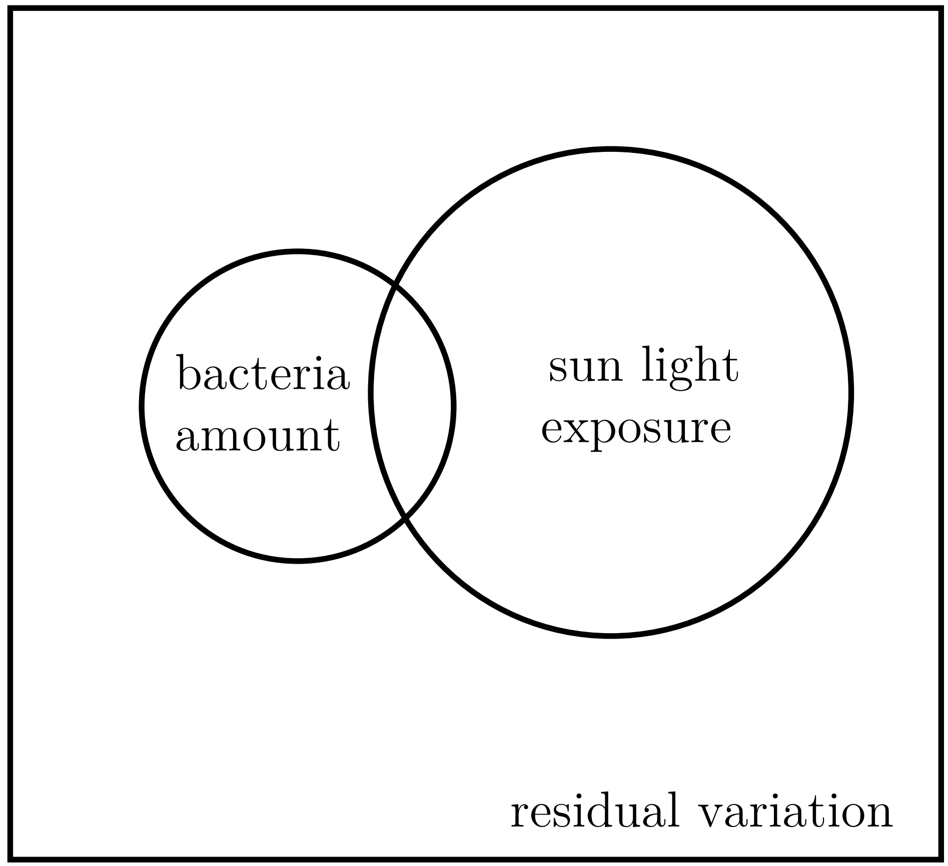
Units attached - cm / % exposure

What do model slopes represent?

Model slopes - represents the difference in predicted value of Y (plant height) for each one unit difference in bacteria amount if amount of sun is kept constant (i.e., as if plants were exposed to the same amount of sun light).

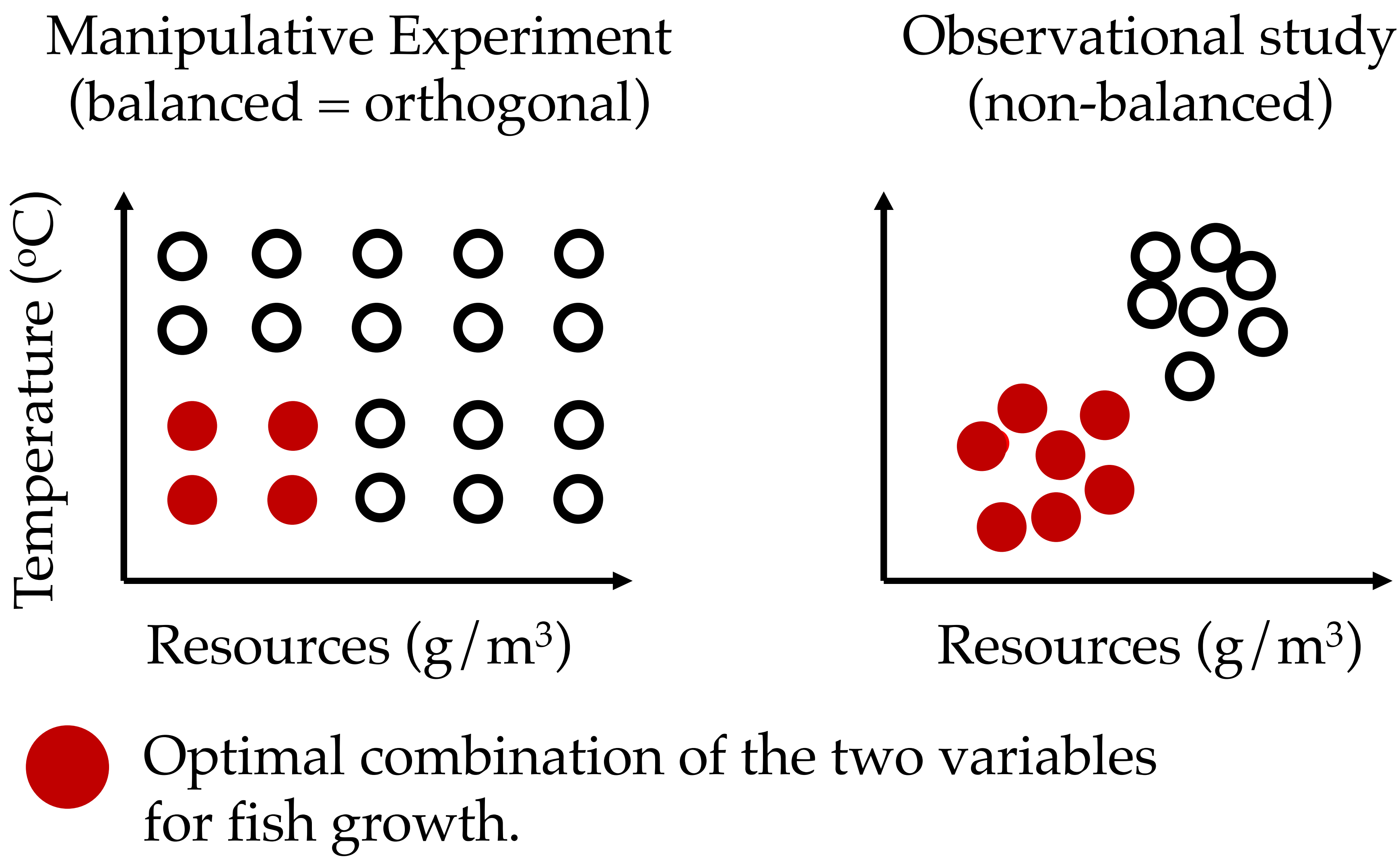
To do that, we use partial slopes – this is important because continuous predictors will rarely be orthogonal and as such we can't assign its effects to one or the other predictor.

Total variation in plant height =

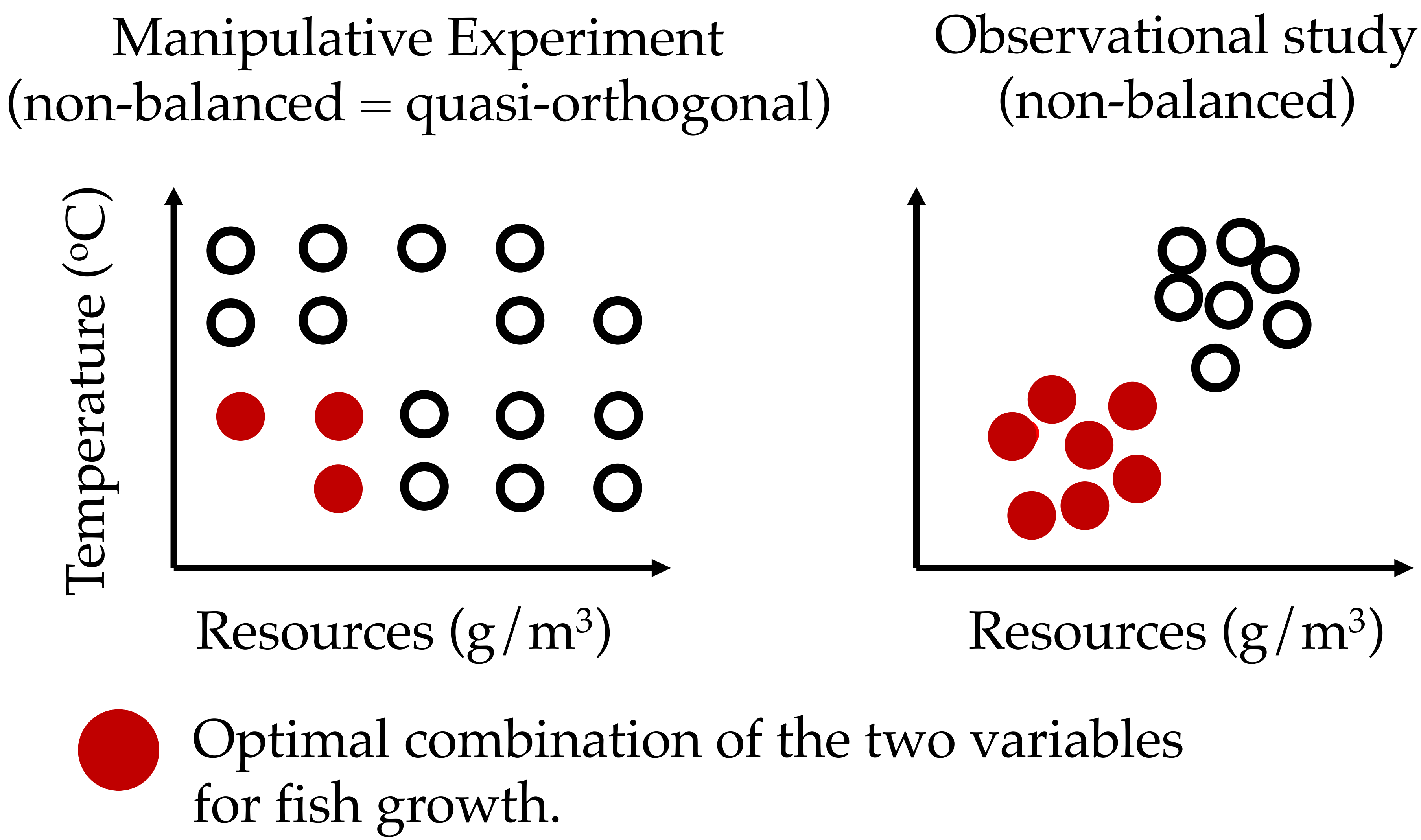


SOUNDS FAMILIAR

Experimental (likely close to orthogonal) versus observational (likely non-orthogonal) approaches.



Experimental (likely close to orthogonal) versus observational (likely non-orthogonal) approaches.



The properties of a regression model (let's use a small simulation)

Regression estimation (based on a sample) of the true population regression involves assumptions.

These assumptions are necessary so that the sample model is an unbiased estimate of the true population model; and that the tests involved have correct behaviour (e.g., Type I error rates = selected alpha).

A word on simulations *versus* math!

The properties of a regression model (let’s use a small simulation)

$$Y = 42\text{cm} + \mathbf{2.3}\text{X}_1 + \mathbf{11}\text{X}_2 + e$$

e residual error assumed to be $N(0, \sigma^2)$

Let’s start with a really large sample size

```

n = 1000000
constant = 42
X1 = rnorm(n,1000,10)
X2 = rnorm(n,40,4)
error = rnorm(n,0,10)

Y = constant + 2.3*X1 + 11*X2 + error

```

The properties of a regression model
(let's use a small simulation)

$$Y = 42\text{cm} + \mathbf{2.3}X_1 + \mathbf{11}X_2 + e$$

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error = rnorm(n,0,10)

Y = constant + 2.3*X1 + 11*X2 + error
```

Model results from simulated data
(large sample size, more accuracy)

```
> lm(Y~X1+X2)

Call:
lm(formula = Y ~ X1 + X2)

Coefficients:
(Intercept)           X1           X2
     42.687        2.299       10.998
```

The properties of a regression model
(let’s use a small simulation)

Let’s reduce sample size

The properties of a regression model (let's use a small simulation)

$$Y = 42\text{cm} + \mathbf{2.3}\text{X}_1 + \mathbf{11}\text{X}_2 + e$$

e residual error are assumed to be $N(0, \sigma^2)$

```r

n = 30
constant = 42
X1 = rnorm(n,1000,10)
X2 = rnorm(n,40,4)
error = rnorm(n,0,10)

Y = constant + 2.3*X1 + 11*X2 + error

```

Model results from simulated data
(smaller sample size, less accuracy; compare with previous example)

```
> lm(Y~X1+X2)

Call:
lm(formula = Y ~ X1 + X2)

Coefficients:
(Intercept)           X1           X2
    247.123        2.076       11.322
```

The properties of a regression model -

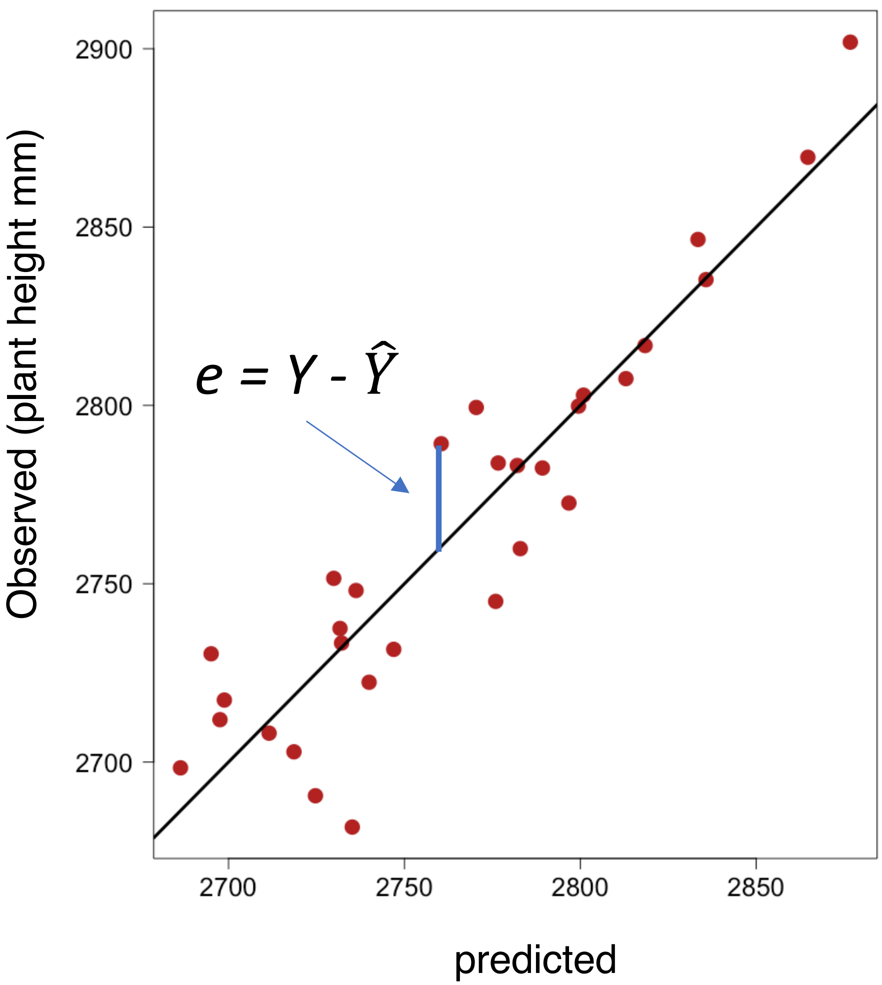
Predicted and residual variation

Understanding predicted values and residuals

$$Y = 247.12 + 2.08X_1 + 11.32X_2 + e$$

$$\hat{Y} = 247.12 + 2.08X_1 + 11.32X_2$$

$$e = Y - \hat{Y}$$

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Understanding predicted values and residuals

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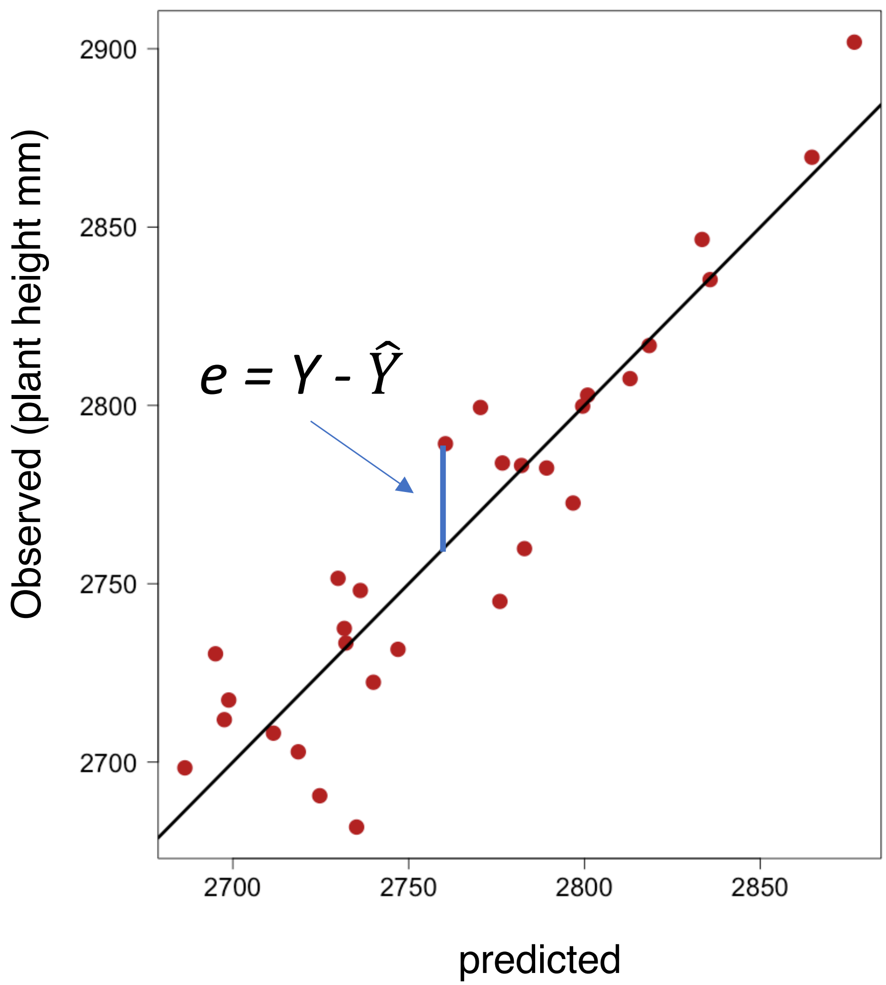
$$\hat{Y} = 247.12 + 2.08X_1 + 11.32X_2$$

$$e = Y - \hat{Y}$$

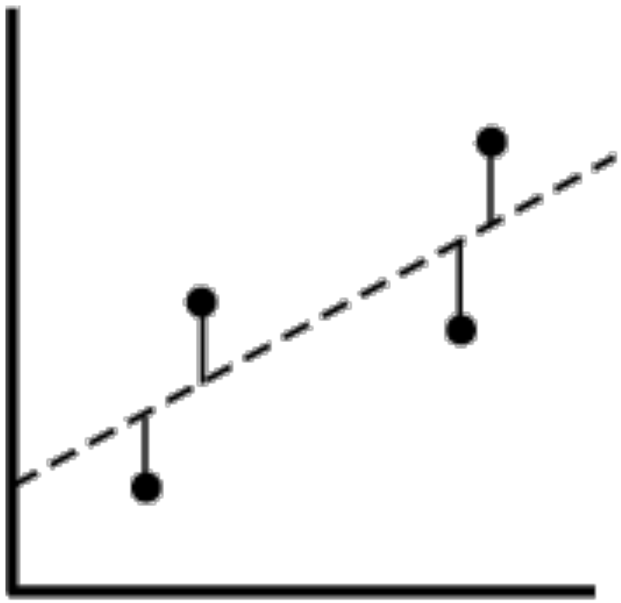


	Y	$\hat{Y}$	e
1	2708.110	2711.535	-3.4253070
2	2783.158	2782.092	1.0661086
3	2835.268	2835.723	-0.4548141
4	2772.625	2796.753	-24.1282583
5	2722.375	2739.964	-17.5887528
6	2748.106	2736.255	11.8513100
7	2759.842	2782.933	-23.0909896
8	2869.578	2864.679	4.8993415
9	2816.781	2818.402	-1.6209332
10	2698.379	2686.358	12.0206930
11	2901.853	2876.740	25.1125353
12	2690.559	2724.710	-34.1513236
13	2717.386	2698.825	18.5610439
14	2711.887	2697.578	14.3091974
15	2730.354	2695.064	35.2899672
16	2846.528	2833.441	13.0866948

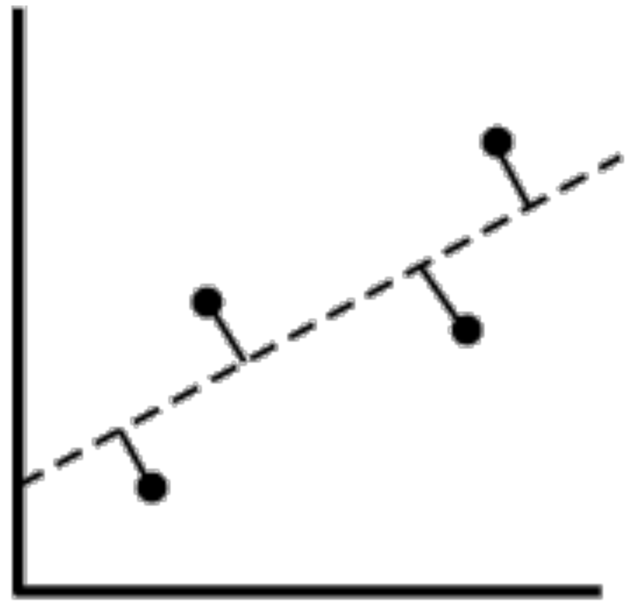
etc (n = 30)

Understanding predicted residuals

multiple regression assumes vertical offsets (residuals)



vertical offsets



perpendicular offsets

Residuals for Type I regression
Error in Y but not in X

Residuals for Type II regression
Error in both Y and X

Type I and III sum-of-squares

Type II sum-of-squares

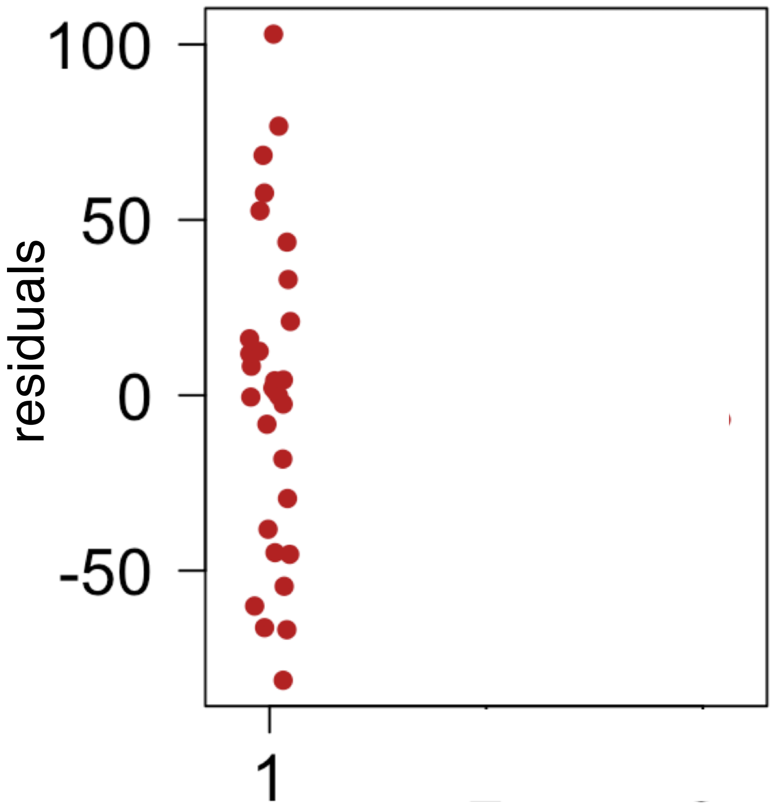
meaningful predictors reduce variance of residuals

A small fictional example to facilitate understanding of what regression coefficients mean!

$$Y = \mathbf{42cm} + \beta_1 X_1 + \beta_2 X_2 + e$$

Y is plant height (cm)
X_1 is amount of bacteria in the soil (count per ml)
X_2 is amount of plant exposure to sun light (% exposure)

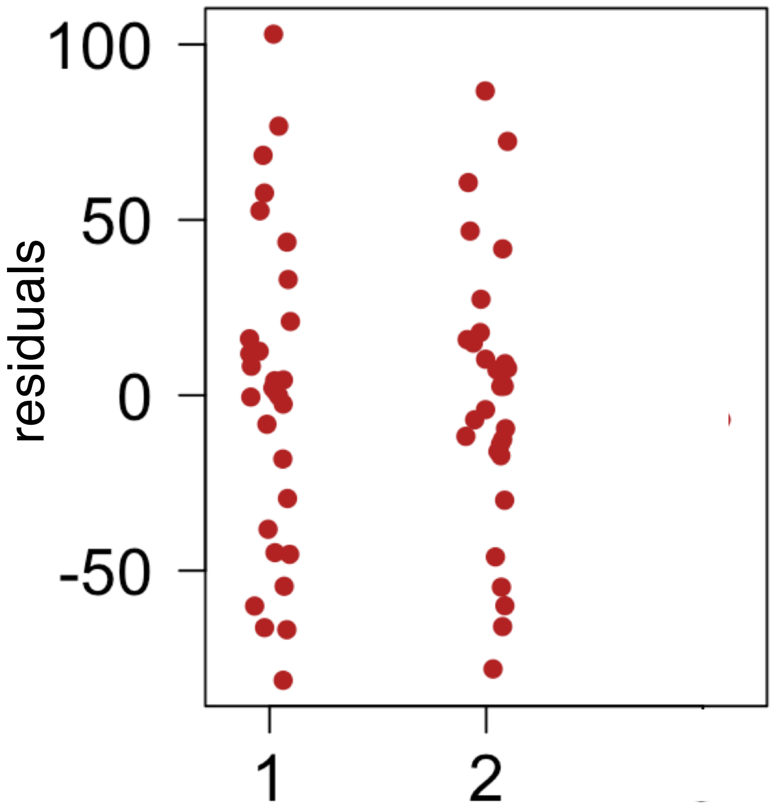
meaningful predictors reduce variance of residuals



$$[1]\ \hat{Y} = \bar{Y}$$

$$e = Y - \hat{Y}$$

meaningful predictors reduce variance of residuals

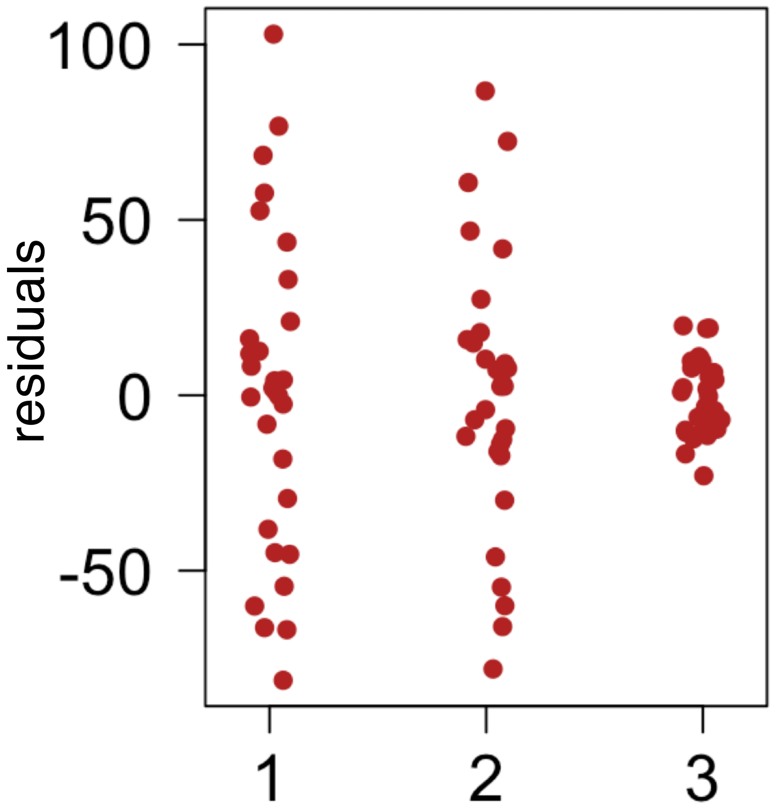


$e = Y - \hat{Y}$

$$[1]\ \hat{Y} = \bar{Y}$$

$$[2]\ \hat{Y} = 247.12 + 2.65X_1$$

meaningful predictors reduce variance of residuals (i.e., uncertainty)



$e = Y - \hat{Y}$

$$[1]\ \hat{Y} = \bar{Y}$$

$$[2]\ \hat{Y} = 247.12 + 2.65X_1$$

$$[3]\ \hat{Y} = 247.12 + 2.08X_1 + 11.32X_2$$

The properties/assumptions of a regression model

Linearity assumption

(big one)

population regression

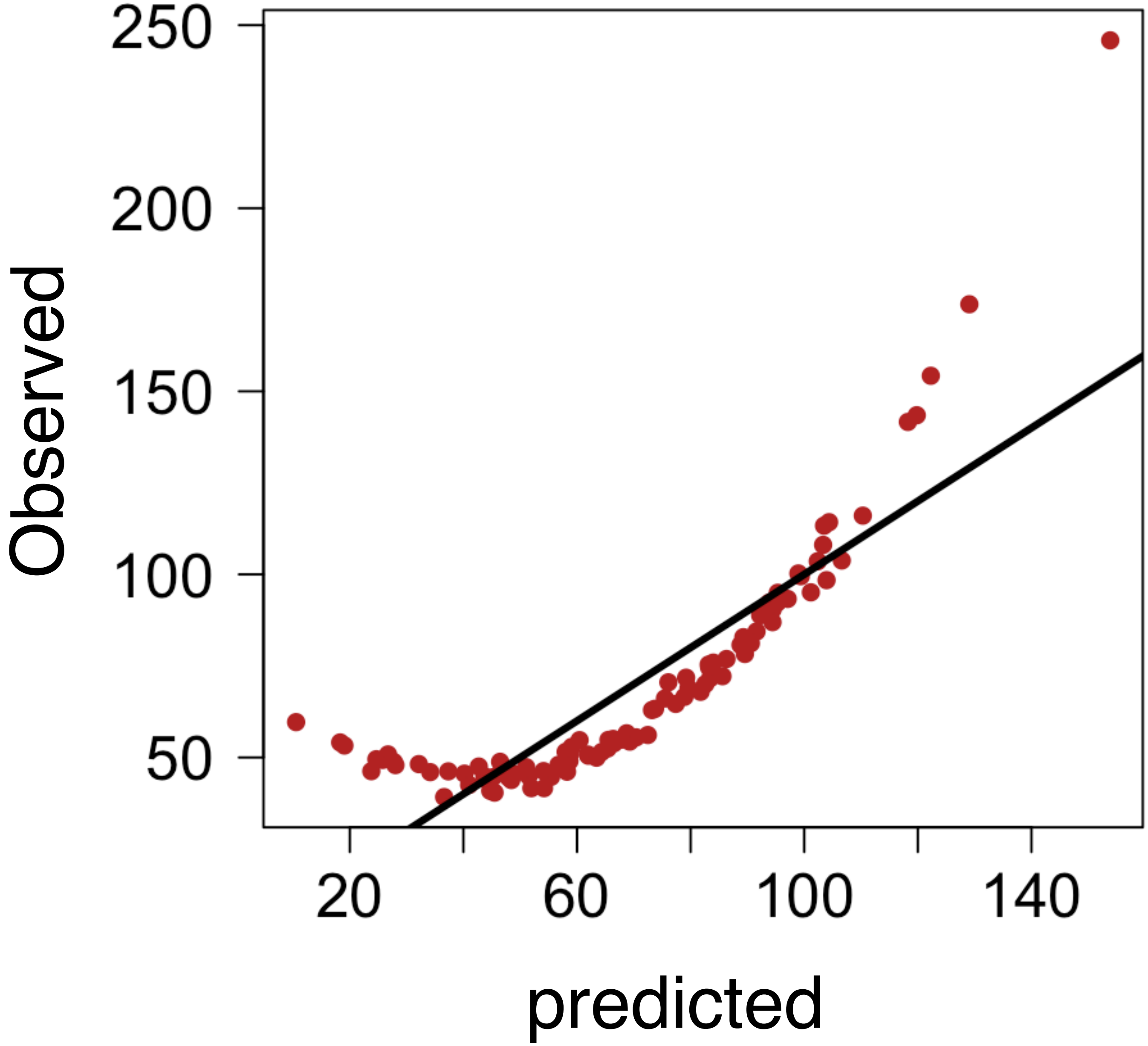
$$Y = 42 + 2.3X_1 + 11X_2^2 + e$$

```r

n = 100
constant = 42
X1 = rnorm(n,1,1)
X2 = rnorm(n,1,1)
error = rnorm(n,0,1)
Y = constant + 2.3*X1 + 11*X2^2 + error

```

sample regression - linear relationship assumed



population regression

$$Y = 42 + 2.3X_1 + 11X_2^2 + e$$

sample regression

```
Call:
lm(formula = Y ~ X1 + X2)
```

treated as linear

```
Coefficients:
(Intercept)           X1           X2
    42.8848      -0.7586      25.6188
```

$$Y = 42 - 0.76X_1 + 25.62X_2$$

population regression

$$Y = 42 + 2.3X_1 + 11X_2^2 + e$$

sample regression (non-linear regression)

```
> lm(Y~X1+I(X2^2))
```

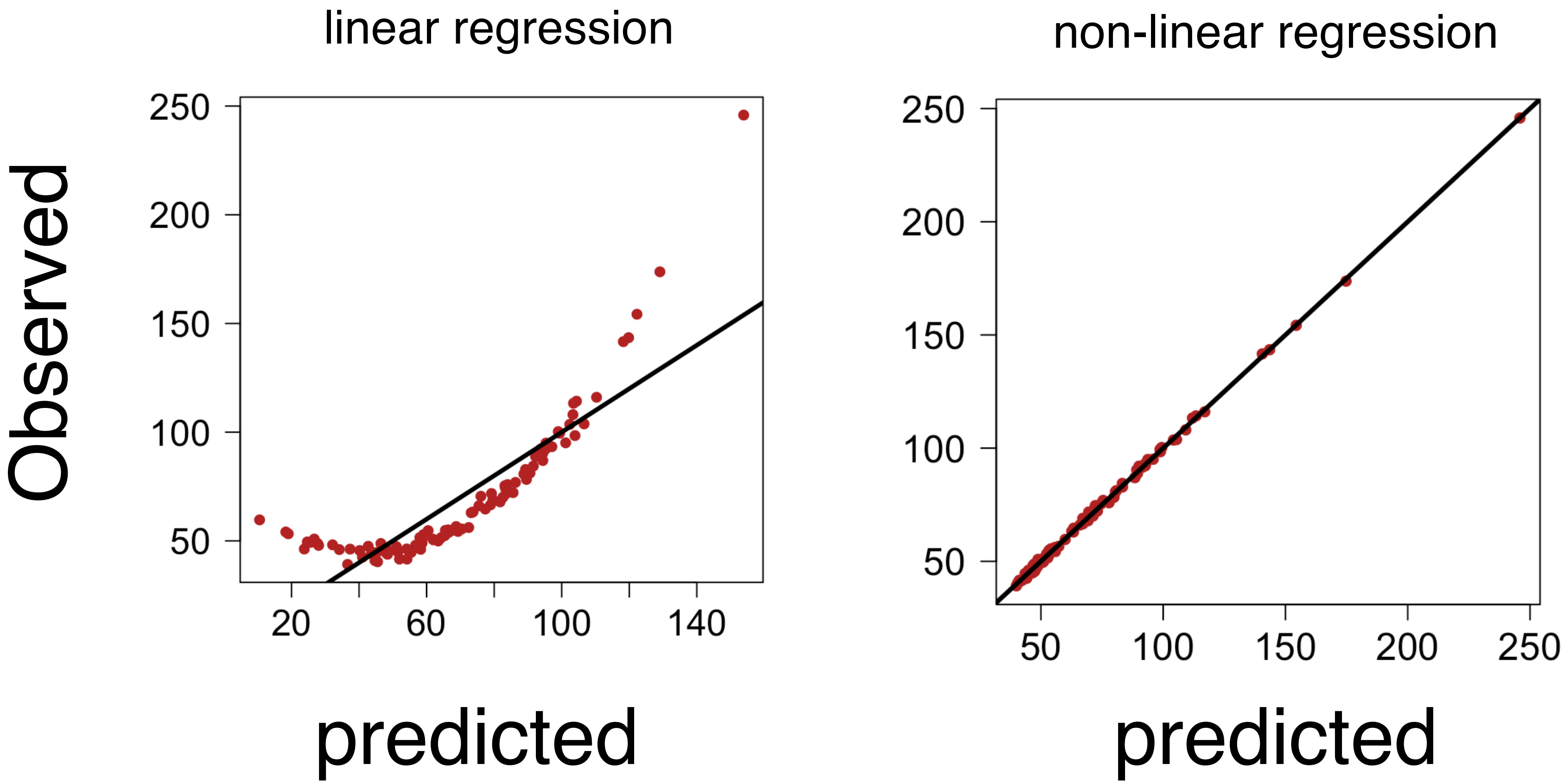
treated as non-linear

```
Call:
lm(formula = Y ~ X1 + I(X2^2))

Coefficients:
(Intercept)           X1      I(X2^2)
      42.17         2.20        10.98
```

$$Y = 42 + 2.2X_1 + 11X_2^2$$

$$Y = 42 + 2.2X_1 + 11X_2^2 + e$$



Effects of non-linear data on regression

More on multiple regressions and assumptions - Lecture 12